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Ramsey–Paris–Harrington Numbers for Graphs

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Ramsey–Paris–Harrington numbers are studied for the special case of graphs, giving reasonably close asymptotic upper and lower bounds for their growth.

INTRODUCTION

J. Paris and L. Harrington have shown in [3] that a certain variant of Ramsey's Theorem cannot be proved in first-order arithmetic although it is in fact a true statement (easily provable in Zermelo–Fraenkel set theory). In [1] P. Erdős and this author gave some bounds for the rates of growth of the functions associated with the exponent two case of this variant. The present paper gives further results in this direction for the even more special case of graphs. In particular we reduce the gap between the lower and upper bounds given in [1] for this case.

Our notation conforms with that of [1] except that we use the terminology of graph theory, rather than of colorings of sets of pairs. Briefly (sets of) positive integers will denoted by (upper, resp.) lowercase Latin letters, $|X|$ will denote the cardinality of X, $\min X$ the minimum element of X, and $[a, b]$ the interval $\{n \mid a \leqslant n \leqslant b\}$. The logarithm function is taken base 2.

Recall that the ordinary Ramsey number $r(m, n)$ is defined to be the smallest integer r such that $r \to (m, n)^2$, that is, for any graph $G = (V, E)$ with $|V| \geqslant r$ there exists $H \subseteq V$ such that either $|H| \geqslant m$ and every two vertices in H are connected by an edge in E (H is *complete*), or $|H| \geqslant n$ and no two vertices in H are so connected (H is *independent*). It is well known (see [2]) that $r(m, n) \leqslant \binom{m+n-2}{m-1}$ for $m, n \geqslant 2$, from which it follows that $r(k-1, m) \leqslant k^{m-1} - k^{m-2}$ for $2 \leqslant m \leqslant k$.

The Paris–Harrington variant of the Ramsey relation is obtained by replacing one or both of the conditions $|H| \geqslant m$ and $|H| \geqslant n$ by the condition that H be *relatively large*, by which is meant $|H| \geqslant \min H$ and $|H| \geqslant 3$. Thus

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$V \to (m, *)^2$ is defined to mean that for any graph $G = (V, E)$ there is a set of vertices $H \subseteq V$ such that either $|H| \geqslant m$ and H is complete or H is relatively large and independent. $V \to (*, n)^2$ and $V \to (*, *)^2$ are defined analogously. For a fuller introduction to the background of this relation, see [1] and [3].

The objective of this paper is to estimate the rate of growth of the following Ramsey–Paris–Harrington functions:

$$R(k; m) = \text{smallest } n \text{ such that } [k, n] \to (m, *)^2$$
$$R(k) = \text{smallest } n \text{ such that } [k, n] \to (*, *)^2.$$

Observe that $R(k; m) \geqslant r(k, m) + k - 1$ since any relatively large subset of $[k, n]$ must have cardinality $\geqslant k$. Also $R(k) \geqslant R(k; k)$ for the same reason. The main conclusions of this paper are as follows.

THEOREM 1. *Exact values and bounds for some concrete cases are*:

	$k=1$	2	3	4	5
$R(k; 3)$	6	8	11	18	25
$R(k; 4)$	15	17–23			
$R(k)$	6	8	13	$\leqslant 279$	

THEOREM 2. (i) *$R(k; 3)/r(k, 3) \to 1$ as $k \to \infty$.*

(ii) *$R(k; 4) = O(r(2k, 4))$.*

The proof of our main result (Theorem 4) depends on the following recursive bounds which are interesting in their own right.

LEMMA 3. (i) *$R(k; m+n) \geqslant R(R(k; m); n+1)$.*

(ii) *$R(k; m+n) \leqslant R(R(k; m)^{m+n-1}; n)$ for $m, n \geqslant 2$ and $k \geqslant 3$.*

(iii) *$R(k) \leqslant R(k+1; 2k-3)$ for $k \geqslant 3$.*

THEOREM 4. *There exist constants α, β, $N > 0$ such that for all $m \geqslant 3$ and $k \geqslant N$*

(i) $k^{2^{\alpha m}} < R(k; m) < k^{2^{\beta m}}$

(ii) $k^{2^{\alpha k}} < R(k) < k^{2^{2^{\beta k}}}$.

The upper bounds in Theorem 4 are really the *raison d'être* of this paper, as they represent the main improvement over the bounds on $R(k)$ obtained in [1], where it was proved that $R(k) < k^{2^{2^{k \log k}}}$. Here the term $\log k$ has been replaced by a constant β, thus closing a gap between lower and upper bounds. By careful calculation one can show that the techniques of this paper yield $\beta = 1.471$ (even smaller in the bounds for $R(k; m)$, depending on m). But for brevity of exposition I will settle here for a β that is twice as large. Interested readers are invited to send for details of the more careful calculation. (But see also the hint in the proof here, which may suffice.)

The key idea for the proof of lower bounds in Theorem 4 is implicit in [1, Theorem 5], due to Erdős, and boils down to the recurrence relation (i) of Lemma 3. By iterating with $n-2$ as in [1], one obtains the lower bound 4(ii) for any $\alpha < 0.5$. Here we iterate with $n=3$ to squeeze out the conclusion for any $\alpha < (\log 3)/3 = 0.528+$. (But α must be smaller in 4(i), depending on m.) Could this be improved to $\alpha < (\log e)/e = 0.530+$ by "iterating with $n = e$"? The proof of Theorem 1 will be omitted except for one graph establishing $R(5; 3) \geqslant 25$. (The other lower bounds are easier, and the upper bounds are tedious.) Note that the upper bound $R(4) \leqslant 279$ improves on $R(4) \leqslant 687$ proved in [1]. Interested readers can write to me for a full proof of Theorem 1.

Theorem 2(ii) could be strengthened to $R(k; 4) = O(r(k, 4))$ if it is true that $r(2k, 4) = O(r(k, 4))$, which seems likely but does not appear in the literature nor could I derive it. Even without this improvement it follows that $R(k; 4) = O(k^3/(\log k)^2)$, the same upper bound as for $r(k, 4)$. This is quite surprising when one compares the lower bound of Theorem 4(i) with the upper bound k^{m-1} for $r(k, m)$. Just how long does one have to wait for the asymptotics to set in? The methods of this paper show that $R(k; 6)$ is larger than $r(k, 6)$ by at least a factor of k, but it remains an intriguing open question whether $R(k; 5) = O(r(ck, 5))$ for some $c > 0$. The methods of this paper only give $R(k; 5) < k^8$, while $r(k, 5) < k^4$.

In the course of the proofs we will make use of the following terminology and general observations. A graph $G = (X, E)$ will be called *$(m, *)$-good* provided it contains no complete set of vertices of cardinality m and no relatively large independent set. Given a vertex $p \in X$, we let $G_p = \{x \in X \mid \{p, x\} \in E\}$ and let $I_p = \{x \in X \mid \{p, x\} \notin E$ and $x \neq p\}$. Thus X is a disjoint union $\{p\} \cup G_p \cup I_p$. In particular if $X = [k, n]$, then $n = k + |G_p| + |I_p|$. Now assume G is $(m, *)$-good. Then one can see that $I_p \not\to (m, p-1)^2$, since if H were an independent set of $p-1$ vertices in I_p, then $\{p\} \cup H$ would be a relatively large independent set. Hence $|I_p| < r(m, p-1)$. Similarly $G_p \not\to (m-1, *)^2$, since a complete set of $m-1$ vertices in G_p would all be joined with p. In the case $m = 3$ we conclude that G_p must be independent set, whence $|G_p| < \min G_p$. We summarize this in the following

LEMMA 5. *Suppose $G = (X, E)$ is an $(m, *)$-good graph and $p, q \in X$, $p \neq q$. Then*

(i) $n = k + |I_p| + |G_p|$,

(ii) $|I_p| \leqslant r(m, p-1) - 1$,

(iii) $G_p \nrightarrow (m-1, *)^2$ ($|G_p| < \min G_p$ *if* $m = 3$),

(iv) $|I_p \cap G_q| \leqslant r(m-1, p-1) - 1$ ($= p - 2$ *if* $m = 3$).

The following lemma is also useful.

LEMMA 6. *If* $[k, n] = A \cup B_1 \cup B_2 \cup \cdots \cup B_t$ *where* $|A| \leqslant a$ *and the* B_i *are not relatively large, then*

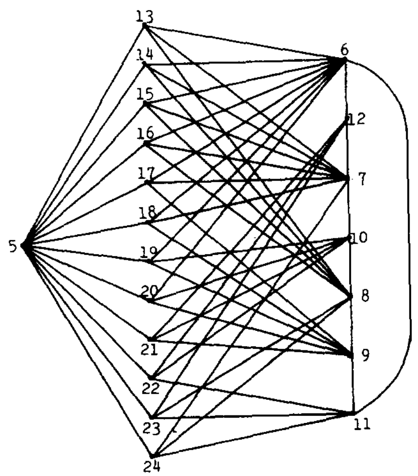
$$n \leqslant 2^t(k + a - 1).$$

Proof. Let $b_i = \min B_i$ and assume w.l.o.g. that $b_1 < b_2 < \cdots < b_t$. Show by induction on i that $b_i \leqslant 2^{i-1}(k + a - 1) + 1$ whence $|B_i| \leqslant 2^{i-1}(k + a - 1)$, using the fact that $b_{i+1} \leqslant k + |A| + \sum_{1 \leqslant j \leqslant i} |B_j|$. ∎

Now to the proofs of the main theorems. As noted, the proof of Theorem 1 will be omitted, except for Fig. 1, which demonstrates that $R(5; 3) \geqslant 25$.

Proof of Theorem 2(i). I show that $R(k; 3) \leqslant r(k-1, 3) + 5k - 7$ for all $k \geqslant 3$, from which the conclusion follows.

Suppose $G = ([k, n], E)$ is a $(3, *)$-good graph. Let $A = G_k$ and $B = I_k$. Then by Lemma 5, $n = k + |A| + |B|$, $|A| < \min A$, and $|B| \leqslant r(k-1, 3) - 1$. If $\min A < 2k$, then $|A| \leqslant 2k - 2$, whence $n \leqslant 3k - 3 + r(k-1, 3) < r(k-1, 3) + 5k - 7$, as desired. So assume $\min A \geqslant 2k$. Then $[k+1, 2k-1] \subseteq B$ and we can find $k + 1 \leqslant p < q \leqslant 2k - 1$ with $p \in G_q$ (else $[k, 2k-1]$ is independent). Now for any $x \in B$ we have $|I_x \cap A| \leqslant x - 2$, by Lemma 5(iv), hence $|G_x \cap A| \geqslant |A| - (x - 2)$. Now $G_p \cap G_q = \varnothing$ (else we get a triangle), so $|A| \geqslant |A \cap G_p| + |A \cap G_q| \geqslant$



vertex	adjacent to
5	13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24
6	11, 12, 13, 14, 15, 16 17, 18, 19, 20
7	10, 12, 13, 14, 15, 16, 17, 18, 24
8	9, 10, 13, 14, 15, 16, 23, 24
9	11, 17, 18, 19, 20, 21
10	19, 20, 21, 22
11	22, 23, 24
12	21, 22, 23

FIG. 1. $[5, 24] \nrightarrow (3, *)^2$, i.e. $R(5; 3) \geqslant 25$.

$2|A|-(p+q-4)$, whence $|A| \leqslant p+q-4 \leqslant 4k-7$. Therefore $n=k+|A|+|B| \leqslant 5k-7+r(k-1,3)-1$. ∎

Proof of Theorem 2(ii). If show that $R(k;4) \leqslant 2r(2k-3,4)+2r(2k-2,3)+2k-1$.

Suppose $G=([k,n],E)$ is a $(4,*)$-good graph. Then $[k,2k-1]$ is not independent, so pick $k \leqslant p < q \leqslant 2k-1$ with $p \in G_q$. Then $[k,n]=\{p,q\} \cup I_p \cup (G_p \cap I_q) \cup (G_p \cap G_q)$. We have $I_p \nrightarrow (4,p-1)^2$, so $|I_p| \leqslant r(p-1,4)-1 \leqslant r(2k-3,4)-1$. Also $G_p \cap I_q \nrightarrow (3,q-1)^2$, so $|G_p \cap I_q| \leqslant r(2k-2,3)-1$. Finally, $G_p \cap G_q$ must be independent, hence not relatively large. So by Lemma 6 we have

$$\begin{aligned} n &\leqslant 2(k+2+|I_p|+|G_p \cap I_q|-1) \\ &\leqslant 2r(2k-3,4)+2r(2k-2,3)+2k-2 \end{aligned}$$

as required. ∎

Proof of Lemma 3(i). Let $w=R(k;m)$ and $z=R(w;n+1)$. Let $G_1=([k,w-1],E_1)$ be an $(m,*)$-good graph and let $G_2=([w,z-1],E_2)$ be an $(n+1,*)$-good graph. Let $G=([k,z-1],E)$ be the union of G_1 and G_2 plus edges joining each vertex in G_1 to every vertex in G_2. Let $H \subseteq [k,z-1]$. If H is complete, then $|H|=|H \cap [k,w-1]|+|H \cap [x,z]| \leqslant (m-1)+(n+1-1) < m+n$. If H is independent, then $H \subseteq [k,w-1]$ or $H \subseteq [w,z-1]$, so H is not relatively large. Thus G is $(m+n,*)$-good, so $R(k;m+n) \geqslant z$. ∎

Proof of Lemma 3(ii). Let $w=R(k;m)$, $u=w^{m+n-1}$, and $z=R(u;n)$.

Suppose $G=([k,z],E)$ were $(m+n,*)$-good. Since $w=R(k;m)$ and there is no relatively large independent set $H \subseteq [k,w]$, we can find $k \leqslant p_1 < p_2 < \cdots < p_m \leqslant w$ with $\{p_1,\ldots,p_m\}$ complete. Now generalizing the technique used to prove Theorem 2(ii), we let $A_0=I_{p_1}$, $A_1=G_{p_1} \cap I_{p_2}$, $A_2=G_{p_1} \cap G_{p_2} \cap I_{p_3}$, etc., up to $A_m=G_{p_1} \cap \cdots \cap G_{p_m}$. Then for $i<m$ we have $A_i \nrightarrow (m+n-i,p_{i+1}-1)^2$, otherwise either some $\{p_1,p_2,\ldots,p_i,x_{i+1},\ldots,x_{m+n}\}$ would be complete or some $\{p_{i+1}\} \cup H$ would be relatively large and independent. Hence $|A_i| < r(m+n-i,p_{i+1}-1) \leqslant r(m+n-i,w-1) \leqslant w^{m+n-i-1}-w^{m+n-i-2}$ for $i<m$. Now

$$[k,z]=\{p_1,\ldots,p_m\} \cup A_0 \cup A_1 \cup \cdots \cup A_m,$$

whence

$$\begin{aligned} |[k,z]-A_m| &\leqslant m+\sum_{i=0}^{m-1} |A_i| \leqslant m+\sum_{i=0}^{m-1} w^{m+n-i-1}-w^{m+n-i-2} \\ &= m+w^{m+n-1}-w^{n-1} \leqslant u-k \\ &= |[k,z]-[u,z]|. \end{aligned}$$

It follows that $|A_m| \geqslant |[u, z]|$ and more importantly there is a one-one monotone mapping f: $[u, z] \to A_m$ with $f(x) \leqslant x$ for all $x \in [u, z]$. Define a graph $G' = ([u, z], E')$ by $\{x, y\} \in E' \Leftrightarrow \{f(x), f(y)\} \in E$. Now $[u, z] \to (n, *)^2$ since $z = R(u; n)$, so let $H \subseteq [u, z]$ witness this. If H is relatively large and independent, then $f''H$ is independent in G and $|f''H| = |H| \geqslant \min H \geqslant f(\min H) = \min f''H$. If $|H| \geqslant n$ and H is complete, then $\{p_1, ..., p_m\} \cup f''H$ is complete in G and has cardinality at least $m + n$. In either case we contradict the goodness of G. ∎

Proof of Lemma 3(iii). Let $z = R(k + 1; 2k - 3)$ and let $G = ([k, z], E)$ be a graph. Define a graph $G' = ([k + 1, z], E')$ by

$$x \in G'_y \Leftrightarrow \begin{cases} x \in G_y & \text{if } \{x, y\} \subseteq G_k \\ x \notin G_y & \text{if } \{x, y\} \subseteq I_k \\ \text{always} & \text{if } |\{x, y\} \cap G_k| = 1. \end{cases}$$

Thus G' is like G on G_k, like the complement of G on I_k, and all vertices in G_k are joined to all vertices in I_k. Let $H \subseteq [k + 1, z]$ witness that $[k + 1, z] \to (2k - 3, *)^2$. If H is independent in G' then H cannot intersect both G_k and I_k. If $H \subseteq G_k$, then H is independent in G. If $H \subseteq I_k$ then H is complete in G. In either case H is still relatively large. On the other hand if H is complete in G' then we look at $H_1 = H \cap G_k$ and $H_2 = H \cap I_k$. Now either $|H_1| \geqslant k - 1$ or $|H_2| \geqslant k - 1$ since $H = H_1 \cup H_2$ and $|H| \geqslant 2k - 3$. In the former case $\{k\} \cup H_1$ is complete and relatively large in G. In the latter, $\{k\} \cup H_2$ is independent and relatively large in G. This proves $[k, z] \to (*, *)^2$, whence $R(k) \leqslant z = R(k + 1; 2k - 3)$. ∎

Proof of Theorem 4, *lower bounds.* Since clearly $r(k, m) < R(k; m)$, we have for any $\varepsilon > 0$, $k^{m-1-\varepsilon} < R(k; m)$ for all sufficiently large k. In particular an $a > 0$ and $N > 0$ can be found so that $k^{2^{a(m-1)}} < R(k; m)$ for $m = 3, 4, 5$ and $k \geqslant N$. Then inductively by Lemma 3(i) $R(k; m + 3) \geqslant R(R(k; m); 4) > (k^{2^{a(m-1)}})^{2^{a3}} = k^{2^{a(m+3-1)}}$. Now pick $\alpha > 0$ so that $\alpha m < a(m - 1)$ for all $m \geqslant 3$ and the proof of (i) is complete.

For (ii), let $\alpha < (\log 3)/3$ be given. Observe that for $m = 4$ and any $\varepsilon > 0$ we have $k^{3-\epsilon} < R(k; 4)$, which means that any $a < (\log 3)/3$ will satisfy $k^{2^{a \cdot 3}} < R(k; 4)$ for all sufficiently large k. By the above inductive argument we have $k^{2^{a(m-1)}} < R(k; m)$ for any $m \equiv 1 \pmod 3$. If a is chosen so that $\alpha < a < (\log 3)/3$, then for all sufficiently large m $k^{2^{\alpha m}} < k^{2^{a(n-1)}} < R(k; n) \leqslant R(k; m)$, where $n = m - 2$, $m - 1$, or m, whichever is $\equiv 1 \pmod 3$. Finally, since any relatively large subset of $[k, R(k)]$ must clearly have cardinality $\geqslant k$, we have $R(k) \geqslant R(k; k) > k^{2^{\alpha k}}$ for all sufficiently large k, as claimed. ∎

Proof of Theorem 4, *upper bounds.* Define a function B recursively as follows: $B(1)=1$ and for $m \geqslant 1$ $B(2m)=(2m-1)B(m)^2$ and $B(2m+1)=2mB(m)B(m+1)$. Define $\beta(m)=(\log B(m))/m$, so that $B(m)=2^{m\beta(m)}$. Let

$$\beta = \sum_{i=0}^{\infty} [\log(3 \cdot 2^i - 1)]/3 \cdot 2^i = 1.471^-.$$

Then the sharpest upper bounds are a consequence of the following two claims:

There is a constant N such that $R(k;m) \leqslant k^{B(m)}$ for all $m \geqslant 3$ and $k \geqslant N$, and (1)

$$\sup_m \beta(m) = \beta. \tag{2}$$

The reader is invited to send for full details. (The sequence $\{\beta(m)\}_{m=1}^{\infty}$ is very peculiar!) For this exposition I will prove (1) and (2) only for m of the form $3 \cdot 2^r$. Theorem 4(i) then follows for general m (but with the constant doubled) by choosing r such that $m \leqslant 3 \cdot 2^r < 2m$, so

$$R(k;m) \leqslant R(k; 3 \cdot 2^r) < k^{2^{2m\beta}}.$$

Proof of (1) for $m = 3 \cdot 2^r$. For $r=0$ we have $B(3)=2$ and $R(k;3)=O(r(k,3))=O(k^2)$ by Theorem 2(i) and known bounds for $r(k,3)$. Inductively for $3 \cdot 2^{r+1} = 2m$ we have

$$R(k; 2m) \leqslant R(R(k;m)^{2m-1}; m) < ((k^{B(m)})^{2m-1})^{B(m)} = k^{B(2m)}$$

by Lemma 3(ii).

To prove (2) for $m = 3 \cdot 2^r$ I show that in fact

$$\beta(3 \cdot 2^r) = \sum_{i=0}^{r} [\log(3 \cdot 2^i - 1)]/3 \cdot 2^i \tag{3}$$

from which (2) follows. For $r=0$ (3) reduces to $\beta(3)=(\log 2)/3$, which is true. Inductively for $3 \cdot 2^r = m$

$$\begin{aligned}\beta(3 \cdot 2^{r+1}) &= (\log B(2m))/2m \\ &= \frac{\log(2m-1) + 2\log B(m)}{2m} \\ &= \frac{\log(2m-1)}{2m} + \beta(m) \\ &= \sum_{i=0}^{r+1} \frac{\log(3 \cdot 2^i - 1)}{3 \cdot 2^i}\end{aligned}$$

as required.

This proves the upper bound in Theorem 4(i). The upper bound in (ii) follows by combining this with Lemma 3(iii):

$$R(k) \leqslant R(k+1; 2k-3) \leqslant (k+1)^{2^{\beta(2k-3)}} < k^{2^{2\beta k}}. \quad \blacksquare$$

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